GENERIC VANISHING INDEX AND THE BIRATIONALITY OF THE BICANONICAL MAP OF IRREGULAR VARIETIES

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ABSTRACT. We prove that any smooth complex projective variety with generic vanishing index bigger or equal than 2 has birational bicanonical map. Therefore, if X is a smooth complex projective variety X with maximal Albanese dimension and non-birational bicanonical map, then the Albanese image of X is fibred by subvarieties of codimension at most 1 of an abelian subvariety of $\operatorname{Alb} X$.

1. Introduction

In the study of smooth complex algebraic varieties, the natural maps provided by the holomorphic forms defined in the variety, have a special importance. For example, the invertible sheaf ω_X of differential n-forms (where n is the dimension of X) produces a map to a projective space, known as the canonical map. The multiples of this canonical sheaf $\omega_X^{\otimes m}$ produce in this way the pluricanonical maps.

$$\varphi_m : X \dashrightarrow \mathbb{P}^N = \mathbb{P}(H^0(X, \omega_X^{\otimes m})^\vee).$$

When φ_m gives a birational equivalence between X and its image, we will simply say that φ_m is birational. We say that X is of general type if for some $m > 0$ the rational map φ_m is birational.

For example, the curves of general type are those of genus $g \geq 2$. The tricanonical map φ_3 is always birational for such curves and the bicanonical φ_2 is also birational once that $g \geq 3$. Moreover, the canonical map is birational as soon as the curve is non-hyperelliptic.

For surfaces, Bombieri [Bo] have given sharp numerical conditions for the birationality of φ_m for $m \geq 3$. The bicanonical map has revealed to be more complicated and has studied by many algebraic geometers. In fact, the surfaces with irregularity $q(S) \leq 1$ and $\chi(S, \omega_S) = 1$ are not completely understood and there is no classification about which ones have birational φ_2. For a modern review of the state of the art in the surface case, we refer to [BCP, Thm. 8].

For higher dimensions not many results are known in general. Nevertheless, the example of the bicanonical map on surfaces shows that for small irregularity $q(X) = h^0(X, \Omega^1_X)$, the classification becomes more difficult. For complex varieties, recall that the differential 1-forms give rise to the Albanese map

$$\operatorname{alb} : X \to \operatorname{Alb} X = H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z}).$$

from X to an abelian variety of dimension $q(X) = h^0(X, \Omega^1_X)$. We say that X is *irregular* if, and only if, $\operatorname{Alb} X$ is not trivial, i.e. $q(X) > 0$. And we say that X is of *maximal Albanese dimension (m.A.d)* if, and only if, the Albanese map $\operatorname{alb} : X \to \operatorname{Alb} X$ is generically finite onto its image.

It turns out that some properties of m.A.d varieties seem to behave independently of the dimension and, indeed, Chen-Hacon showed that this is the case for their pluricanonical maps.

Theorem (Chen-Hacon. [CH2])**.**

(a) X m.A.d and $\chi(\omega_X) > 0 \Rightarrow X$ is of general type, furthermore, φ_3 is birational.
(b) X m.A.d $\Rightarrow \varphi_6$ is the stable pluricanonical map.

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For φ_2, we cannot expect to use $\chi(\omega_X)$ to control directly its birationality. For example, if C is a curve of genus 2, then the bicanonical map of the product $C \times Y$ is never birational. In fact, it is clear that any variety that admits a fibration whose general fibre has non-birational φ_2 will have a non-birational bicanonical map. This should be considered, at least at first glance, as the standard case for higher dimensional varieties.

The following theorem provides geometric constraints for the non-birationality of the bicanonical map (see Theorem 5.2).

Theorem A. *Let X be a smooth projective complex variety of maximal Albanese dimension such that the bicanonical map is not birational. Then, the Albanese image of X is fibred by subvarieties of codimension at most 1 of an abelian subvariety of $\operatorname{Alb} X$. The base of the fibration is also of maximal Albanese dimension.*

That is, X admits a fibration onto a normal projective variety Y with $0 \leq \dim Y < \dim X$, such that any smooth model $\tilde{Y}$ of Y is of maximal Albanese dimension and

$$q(X) - \dim X \leq q(\tilde{Y}) - \dim Y + 1.$$

Hence, when $q(X) > \dim X + 1$ implies the existence of an actual fibration, i.e. $\dim Y > 0$, whose general fibre is mapped generically finite through the Albanese map of X either onto a fixed abelian subvariety of $\operatorname{Alb} X$, or onto a divisor of this fixed abelian subvariety. When $\dim Y = 0$ the theorem simply says that the image of X in $\operatorname{Alb} X$ has codimension at most 1.

In particular, when X does not admit any fibration and $q(X) > \dim X$, there is only one possible case, i.e. X is birationally equivalent to a theta-divisor of an indecomposable principally polarized abelian variety (see [BLNP, Thm. A]). When X does not admit any fibration and $q(X) = \dim X$, there is only one known case of variety of general type and non-birational bicanonical map: a double cover of a principally polarized abelian variety (A, Θ) branched along a reduced divisor $B \in |2\Theta|$. Is this the only case? The answer is affirmative in the case of surfaces due to Ciliberto-Mendes Lopes [CM, Thm 1.1].

To deduce Theorem A it is useful to consider the generic vanishing index introduced by Pareschi–Popa in [PP3, Def. 3.1]

$$\operatorname{gv}(\omega_X) = \min_{i>0} \left\{ \operatorname{codim}_{\operatorname{Pic}^0 X} V^i(\omega_X) - i \right\},$$

where $V^i(\omega_X) = \left\{ \alpha \in \operatorname{Pic}^0 X \,\middle|\, h^i(X, \omega_X \otimes \alpha) > 0 \right\}$. As a consequence of Generic Vanishing Theorem of Green–Lazarsfeld [GL1, Thm. 1], we have that for any irregular variety $1 - \dim X \leq \operatorname{gv}(\omega_X) \leq q(X) - \dim X$.

Moreover, the negative values of $\operatorname{gv}(\omega_X)$ can be interpreted in terms of the dimension of the generic fibre of the Albanese map (see Theorem 3.7) and X is a m.A.d variety if, and only if, $\operatorname{gv}(\omega_X) \geq 0$. Due to the work of Pareschi–Popa [PP3] we can interpret the positive values of $\operatorname{gv}(\omega_X)$ in terms of the local properties of the Fourier-Mukai transform of the structural sheaf (see Theorem 3.3). They have also proved that the positive values of $\operatorname{gv}(\omega_X)$ give a lower bound for the Euler characteristic $\chi(\omega_X)$ (see Theorem 3.4).

Using the generic vanishing index we have the following more synthetic result.

Theorem B. *Let X be a smooth projective complex variety such that $\operatorname{gv}(\omega_X) \geq 2$. Then, the rational map associated to $\omega_X^2 \otimes \alpha$ is birational onto its image for every $\alpha \in \operatorname{Pic}^0 X$.*

Theorem A is deduced from this result by an argument of Pareschi-Popa. On the other hand, this result (see Theorem 5.1) is proved using a birationality criterion (see Lemma 4.2) that is a slight modification of [BLNP, Thm. 4.13].

For curves, $\operatorname{gv}(\omega_C) \geq 2$ is equivalent to $g(C) \geq 3$. For surfaces, $\operatorname{gv}(\omega_S) \geq 2$ is equivalent to suppose that $q(S) \geq 4$ and does not admit an irregular fibration to a curve of genus $\leq q(S) - 3$ (see Example 5.3).

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2. Generalized Fourier-Mukai transform

X will be a smooth projective variety over an algebraically closed field k (from section 3.3 on, we will restrict to $k = \mathbb{C}$). It will be equipped with a morphism $a : X \to A$ to a non-trivial abelian variety A, in particular, X will be irregular. Let $\mathcal{P}$ be a Poincaré line bundle on $A \times \operatorname{Pic}^0 A$. We will denote

$$P_a = (a \times \mathrm{id}_{\operatorname{Pic}^0 X})^* \mathcal{P}, \tag{1}$$

the *induced Poincaré line bundle* in $X \times \operatorname{Pic}^0 A$. When $a = \mathrm{alb}$, the Albanese map of X, then the map alb^* identifies $\operatorname{Pic}^0(\operatorname{Alb} X)$ to $\operatorname{Pic}^0 X$ and the line bundle P_{alb} will be simply denoted by P.

Letting p and q the two projections of $X \times \operatorname{Pic}^0 A$, we consider the left exact functor $\Phi_{P_a}\mathcal{F} = q_*(p^*\mathcal{F} \otimes P_a)$, and its right derived functors

$$R^i\Phi_{P_a}\mathcal{F} = R^i q_*(p^*\mathcal{F} \otimes P_a). \tag{2}$$

Sometimes we will have to consider the analogous derived functor $R^i\Phi_{P_a^{-1}}\mathcal{F}$ as well. By the Seesaw Theorem [Mu, Cor. 6, pg. 54], $\mathcal{P}^{-1} = (1_A \times (-1)_{\operatorname{Pic}^0 A})^*\mathcal{P}$, so

$$R^i\Phi_{P_a^{-1}}\mathcal{F} = (-1_{\operatorname{Pic}^0 A})^* R^i\Phi_{P_a}\mathcal{F} \qquad \text{for any } i. \tag{3}$$

Given a coherent sheaf $\mathcal{F}$ on X, its *i-th cohomological support locus with respect to a* is

$$V_a^i(\mathcal{F}) = \left\{\alpha \in \operatorname{Pic}^0 A \,\middle|\, h^i(\mathcal{F} \otimes a^*\alpha) > 0\right\}$$

Again, when a is the Albanese map of X, we will omit the subscript, simply writing $V^i(\mathcal{F})$. By base change, these loci contain the set-theoretical support of $R^i\Phi_{P_a}\mathcal{F}$, i.e. $\operatorname{supp} R^i\Phi_{P_a}\mathcal{F} \subseteq V_a^i(\mathcal{F})$.

A way to measure the size of all the $V_a^i(\mathcal{F})$'s is provided by the following invariant introduced by Pareschi–Popa.

Definition 2.1 ([PP3, Def. 3.1])**.** Given a coherent sheaf $\mathcal{F}$ on X, the *generic vanishing index* of $\mathcal{F}$ (with respect to a) is

$$\mathrm{gv}_a(\mathcal{F}) := \min_{i>0} \left\{\operatorname{codim}_{\operatorname{Pic}^0 A} V_a^i(\mathcal{F}) - i\right\}.$$

By convention we define $\mathrm{gv}_a(\mathcal{F}) = \infty$, when $V_a^i(\mathcal{F}) = \emptyset$ for every $i > 0$. When a is the Albanese map of X, we will omit the subscript, simply writing $\mathrm{gv}(\mathcal{F})$.

By base change (see [PP3, Lem. 2.1]) it is easy to see that $\mathrm{gv}_a(\mathcal{F})$ can be also defined as the $\min_{i>0}\left\{\operatorname{codim}_{\operatorname{Pic}^0 A} \operatorname{supp} R^i\Phi_{P_a}\mathcal{F} - i\right\}$.

3. Generic vanishing index of the canonical sheaf

3.1. **Relations between $\mathrm{gv}(\omega_X)$ and the Fourier-Mukai transform of $\mathcal{O}_X$.** Here we specialize some general results of Pareschi–Popa [PP3, PP4] to the canonical sheaf of a smooth projective variety of dimension d. Some of these results were previously obtained by Hacon (see [Ha]).

The negative values of the gv-index are related with the vanishing of the lowest cohomologies of the Fourier-Mukai transform of its Grothendieck dual. In the case of ω_X this can be stressed simply as:

Theorem 3.1 ([PP3, Thm. 2.2])**.** *The following are equivalent,*

(a) $\mathrm{gv}_a(\omega_X) \geq -e$ *for* $e \geq 0$*;*
(b) $R^i\Phi_{P_a}\mathcal{O}_X = 0$ *for all* $i \neq d-e, \dots, d$.

Hence, when $\mathrm{gv}_a(\omega_X) \geq 0$, $R^i\Phi_{P_a}\mathcal{O}_X = 0$ for all $i \neq d$, and we usually denote

$$\widehat{\mathcal{O}_X} = R^d\Phi_{P_a}\mathcal{O}_X.$$

Note that, in this case, $H^i(X, \omega_X \otimes a^*\alpha) = 0$ for all $i > 0$ and general $\alpha \in \operatorname{Pic}^0 A$. Therefore, by deformation-invariance of χ, the generic value of $h^0(X, \omega_X \otimes a^*\alpha)$ equals $\chi(\omega_X)$, in particular $\chi(\omega_X) \geq 0$. Since, by base-change, the fibre of $\widehat{\mathcal{O}_X}$ at a general point $\alpha \in \operatorname{Pic}^0 A$ is isomorphic to $H^d(X, a^*\alpha) \cong H^0(X, \omega_X \otimes a^*\alpha^{-1})^*$, the (generic) rank of $\widehat{\mathcal{O}_X}$ is $\operatorname{rk}\widehat{\mathcal{O}_X} = \chi(\omega_X)$.

From Grothendieck-Verdier duality [Co, Thm. 4.3.1] and Theorem 3.1 it follows that,

Corollary 3.2 ([PP4, Rem. 3.13])**.** *If* $\mathrm{gv}_a(\omega_X) \geq 0$ *then* $\mathcal{E}xt^i_{\mathcal{O}_{\operatorname{Pic}^0 A}}((-1_{\operatorname{Pic}^0 A})^*\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 A}) \cong R^i\Phi_{P_a}\omega_X$.

The following result of Pareschi–Popa gives a dictionary between the positive values of $\mathrm{gv}_a(\omega_X)$ and the local properties of the Fourier-Mukai transform of $\widehat{\mathcal{O}_X}$.

Theorem 3.3 ([PP3, Cor. 3.2])**.** *Assume that* $\mathrm{gv}_a(\omega_X) \geq 0$. *Then,*

$$\mathrm{gv}_a(\omega_X) \geq m \ \ \textit{if, and only if,} \ \ \widehat{\mathcal{O}_X} \ \textit{is a } m\textit{-syzygy sheaf.} \tag{4}$$

In particular, $\mathrm{gv}_a(\omega_X) \geq 1$ *is equivalent to* $\widehat{\mathcal{O}_X}$ *being torsion-free and* $\mathrm{gv}_a(\omega_X) \geq 2$ *to* $\widehat{\mathcal{O}_X}$ *being reflexive.*

Using the Evans–Griffith Syzygy Theorem and the previous theorem, Pareschi–Popa obtain the following bound on the Euler holomorphic characteristic that generalizes to higher dimensions the Castelnuovo-de Franchis inequality.

Theorem 3.4 ([PP3, Thm. 3.3])**.** *Assume that* $\mathrm{gv}_a(\omega_X) \geq 0$. *Then,* $\chi(\omega_X) \geq \mathrm{gv}_a(\omega_X)$.

Remark 3.5. *In fact, the theorem of Pareschi–Popa is more general, namely that for any coherent sheaf* $\mathcal{F}$ *if* $\infty > \mathrm{gv}_a(\mathcal{F}) \geq 0$, *then* $\chi(\mathcal{F}) \geq \mathrm{gv}_a(\mathcal{F})$. *As a consequence, we easily obtain that for any non-zero coherent sheaf* $\mathcal{F}$, $\mathrm{gv}_a(\mathcal{F}) \geq 1 \Rightarrow \chi(\mathcal{F}) \geq 1$. *Observe also that if* a *is non-trivial, we always have* $\mathrm{gv}_a(\omega_X) < \infty$.

3.2. **Top Fourier-Mukai transform of the canonical sheaf.** In the case of abelian varieties (or complex torus) the following result is well-known and crucial in the proof of the Mukai Equivalence Theorem [M, Thm 2.2]. We will need it in the proof of Theorem 5.1.

Proposition 3.6 ([BLNP, Prop. 6.1])**.** *If* $a^* : \operatorname{Pic}^0 A \to \operatorname{Pic}^0 X$ *is an embedding, then*

$$R^d\Phi_{P_a}\omega_X \cong k(\hat{0}).$$

3.3. **Generic vanishing theorem of Green–Lazarsfeld.** The name of the gv-index comes from the well-known Generic Vanishing Theorem of Green–Lazarsfeld. As other general vanishing theorems, it requires $\operatorname{char} k = 0$ so from now on we will restrict ourselves to the case $k = \mathbb{C}$. Basically, the following theorem is [GL1, Thm. 1]. The converse implication was proven independently in [LP, Thm. B] and [BLNP, Prop. 2.7].

Theorem 3.7. *For any* $e > 0$, *the following are equivalent:*

(a) the generic fibre of $a\colon X \to A$ *has dimension* e,
(b) $\mathrm{gv}_a(\omega_X) = -e$.

Moreover $\mathrm{gv}_a(\omega_X) \geq 0$ *if, and only if,* $a\colon X \to A$ *is generically finite onto its image.*

In particular, observe that for any irregular variety $1 - \dim X \leq \mathrm{gv}(\omega_X) \leq q(X) - \dim X$.

Remark 3.8. *If* $\mathrm{gv}_a(\omega_X) \geq 0$ *and* $\chi(\omega_X) > 0$, *then* X *is a variety of general type. Indeed, by the previous result* $a : X \to A$ *is generically finite and since* $\chi(\omega_X) > 0$, *we have that* $V^0_a(\omega_X) = \operatorname{Pic}^0 A$, *so by* [CH1, Cor.2.4], $\kappa(X) = \dim X$. *In particular, if* $\mathrm{gv}_a(\omega_X) \geq 1$, *then* X *is of general type.*

3.4. **Subtorus theorem of Green–Lazarsfeld and Simpson.** The following theorem is due to Green and Lazarsfeld [GL2, Thm 0.1] with an important addition due to Simpson [S, §4,6,7].

Theorem 3.9. *Let W an irreducible component of $V^i(\omega_X)$ for some i. Then,*

(a) There exists a torsion point $\beta \in \operatorname{Pic}^0 X$ and a subtorus B of $\operatorname{Pic}^0 X$ such that $W = \beta + B$.
(b) There exists a normal variety Y of dimension $\leq d - i$, such that any smooth model of Y has maximal Albanese dimension and a morphism with connected fibres $f\colon X \to Y$ such that B is contained in $f^* \operatorname{Pic}^0 Y$.

Remark 3.10. *It is useful to recall that the morphism $f\colon X \to Y$ in the second part of the previous theorem, arises as the Stein factorization of the morphism $\pi \circ \mathrm{alb}\colon X \to \operatorname{Pic}^0 W$, where $\pi\colon \operatorname{Alb} X \to \operatorname{Pic}^0 W$ is the dual map of the inclusion $W \subseteq \operatorname{Pic}^0 X$. Hence, the key point of the second part of the theorem is the dimensional bound for Y.*

4. Birationality criterion for maximal Albanese dimension varieties

In this section, we will assume that $a : X \to A$ is a generically finite morphism onto its image, where A is an abelian variety. We introduce another piece of notation.

Notation 4.1. *Let $\mathcal{F}$ be a subsheaf of a line bundle and suppose that $\mathrm{gv}_a(\mathcal{F}) \geq 1$.*

(a) We denote $U_{\mathcal{F}}$, the open subset where $h^0(\mathcal{F} \otimes a^*\alpha)$ has the minimal value, i.e. $\chi(\mathcal{F})$.
(b) Let Z be the exceptional locus of $a : X \to A$, that is $Z = a^{-1}(T)$, where T is the locus of points in A over which the fibre of a has positive dimension.
(c) We define
$$\mathcal{B}_a^{\mathcal{F}}(x) = \{\alpha \in U_{\mathcal{F}} \,|\, x \text{ is a base point of } |\mathcal{F} \otimes a^*\alpha|\}\,.$$
By Remark 3.5, $\chi(\mathcal{F}) \geq 1$. So, by semicontinuity, it makes sense to speak of the base locus of $\mathcal{F} \otimes a^*\alpha$ for all $\alpha \in \operatorname{Pic}^0 A$.

The following lemma is a slight modification of [BLNP, Thm. 4.13] and it is based on [PP1, Prop. 2.12 and 2.13].

Lemma 4.2. *Suppose that $a : X \to A$ is a generically finite morphism onto its image and let $\mathcal{F}$ be a subsheaf of a line bundle such that $\mathrm{gv}_a(\mathcal{F}) \geq 1$ and $R^i a_* \mathcal{F} = 0$ for all $i > 0$. Suppose that for a general $x \in X$,*
$$\operatorname{codim}_{U_{\mathcal{F}}} \mathcal{B}_a^{\mathcal{F}}(x) \geq 2.$$
Then, the rational map associated to the linear system $|\mathcal{F} \otimes L|$ is birational for every line bundle L such that $\mathrm{gv}_a(L) \geq 1$.

Proof. We first compare the Fourier-Mukai transform of $\mathcal{F} \otimes \mathcal{I}_x$ and $\mathcal{F}$.
Claim. Let $x \in X$ be a closed point out of Z. Then $R^i a_*(\mathcal{F} \otimes \mathcal{I}_x \otimes a^*\alpha) = 0$ for $i > 0$. This follows immediately from the exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{I}_x \to \mathcal{F} \to k(x) \to 0 \tag{5}$$

and the hypothesis that $R^i a_* \mathcal{F} = 0$, a is generically finite and $x \notin Z$. Hence, the degeneration of the Leray spectral sequence yields to

$$V_a^i(\mathcal{F} \otimes \mathcal{I}_x) = V^i(a_*(\mathcal{F} \otimes \mathcal{I}_x)). \tag{6}$$

By sequence (5), tensored by $a^*\alpha$, it follows that

$$V_a^i(\mathcal{F} \otimes \mathcal{I}_x) = V_a^i(\mathcal{F}) \qquad \text{for all } i \geq 2. \tag{7}$$

For $i = 1$ we have the surjection $H^1(\mathcal{F} \otimes \mathcal{I}_x \otimes a^*\alpha) \twoheadrightarrow H^1(\mathcal{F} \otimes a^*\alpha)$, that is an isomorphism if, and only if, x is not a base point of $|\mathcal{F} \otimes a^*\alpha|$. In other words $V_a^1(\mathcal{F} \otimes \mathcal{I}_x) \subseteq \mathcal{B}_a^{\mathcal{F}}(x) \cup V_a^1(\mathcal{F})$. Since $\mathrm{gv}_a(\mathcal{F}) \geq 1$, the hypothesis on $\mathcal{B}_a^{\mathcal{F}}(x)$ guarantees that

$$\operatorname{codim} V_a^1(\mathcal{F} \otimes \mathcal{I}_x) \geq 2, \tag{8}$$

for a general $x \in X \setminus Z$. Hence by (6), (7) and (8), $\mathrm{gv}(a_*(\mathcal{F} \otimes \mathcal{I}_x)) \geq 1$. By [PP1, Prop. 2.13], $a_*(\mathcal{F} \otimes \mathcal{I}_x)$ is continuously globally generated (CGG, see [PP1]). Therefore $\mathcal{F} \otimes \mathcal{I}_x$ itself is CGG outside Z (with respect to a). Since the same is true for L, it follows from [PP1, Prop 2.12] that for all $\alpha \in \mathrm{Pic}^0 A$, $\mathcal{F} \otimes L \otimes \mathcal{I}_x$ is globally generated outside Z. So the rational map associated to $|\mathcal{F} \otimes L|$ is birational. □

Remark 4.3. *From the proof we see that if* $\mathrm{codim}_{U_{\mathcal{F}}} \mathcal{B}^{\mathcal{F}}_a(x) \geq 2$ *for every* $x \in X \setminus Z$*, then* $\mathcal{F} \otimes L$ *is very ample out of* Z*, the exceptional locus of* a.

4.1. **Adjoint line bundles.** When $\mathcal{F} = \omega_X$ we will call $U_{\mathcal{F}}$ simply U_0 and $\mathcal{B}^{\omega_X}_a(x)$ simply by

$$\mathcal{B}_a(x) = \{\alpha \in U_0 \,|\, x \text{ is a base point of } \omega_X \otimes a^*\alpha\}\,. \tag{9}$$

Throughout subsections §4.1 and §4.2, we will assume that $\mathrm{gv}_a(\omega_X) \geq 1$.

Proposition-Definition 4.4. *Let* X *be a variety such that* $\mathrm{gv}_a(\omega_X) \geq 1$ *and let* L *be any line bundle on* X *such that* $\mathrm{gv}_a(L) \geq 1$*. Suppose that there exists* $\alpha \in \mathrm{Pic}^0 A$ *such that* $\omega_X \otimes L \otimes a^*\alpha$ *is not birational. Then,*

$$\mathrm{codim}_{X \times U_0} \{(x, \alpha) \in X \times U_0 \,|\, x \textit{ is a base point of } \omega_X \otimes a^*\alpha\} = 1,$$

and its divisorial part is dominant on X *and surjects on* U_0 *via the projections* p *and* q*. We endow this set with the natural subscheme structure given by the image of the relative evaluation map* $q^*(q_*\mathcal{L}) \otimes \mathcal{L}^{-1} \to \mathcal{O}_{X \times U_0}$*, where* $\mathcal{L} = (p^*\omega_X) \otimes P_a)|_{X \times U_0}$ *and we call* $\mathcal{Y}$ *the union of its divisorial components that dominate* U_0*. Let* $\overline{\mathcal{Y}}$ *be its closure in* $X \times \mathrm{Pic}^0 X$*. Then*

- *(a)* X *is covered by the scheme-theoretic fibres of the projection* $\overline{\mathcal{Y}} \to U_0$*, that we will call* F_α*, for* α *varying in* U_0*. By definition, at a* general *point* $\alpha \in U_0$*,* F_α *is the fixed divisor of* $\omega_X \otimes a^*\alpha$*.*
- *(b) For a general* $x \in X$*, the fibre of the projection* $\overline{\mathcal{Y}} \to X$ *is a divisor, that we will call* $\mathcal{D}_x$*. By definition,* $\mathcal{D}_x$ *is the closure of the union of the divisorial components of the locus of* $\alpha \in U_0$ *such that* $x \in \mathrm{Bs}(\omega_X \otimes a^*\alpha)$*.*

Proof. Everything follows from taking $\mathcal{F} = \omega_X$ in Lemma 4.2. The surjectivity of the projection to U_0 is consequence of the Castelnuovo-de Franchis inequality 3.4, i.e. $\chi(\omega_X) \geq \mathrm{gv}_a(\omega_X) \geq 1$. □

4.2. **Decomposition.** In the sequel we will need $a^* : \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$ to be an embedding. However, for simplicity we will go one step further and we will simply suppose that $A = \mathrm{Alb}\, X$. Suppose that we are under the hypotheses of the previous Proposition-Definition and consider a fixed point $\alpha_0 \in U_0$, and the map

$$f_{\alpha_0} \colon U_0 \to \mathrm{Pic}^0 X \qquad \alpha \mapsto \mathcal{O}_X(F_\alpha - F_{\alpha_0}), \tag{10}$$

where F_α is the divisor defined in Proposition-Definition 4.4*(a)*. For $\alpha \in U_0$, all the F_α are algebraically equivalent since they are the fibres of $\overline{\mathcal{Y}} \to U_0$, so the map is well-defined.

The following lemma shows that this map induces a decomposition of $\mathrm{Pic}^0 X$ and that the divisors F_α move algebraically along a non-trivial factor of $\mathrm{Pic}^0 X$. Although the proof is basically the same as [BLNP, Lem. 5.1], we do not require $V^1(\omega_X)$ to be a finite set, but only a proper subvariety.

Lemma 4.5. *The map defined in* (10)*, induces an homomorphism* $f : \mathrm{Pic}^0 X \to \mathrm{Pic}^0 X$ *such that,*

- *(a)* $f^2 = f$ *and* $\mathrm{Pic}^0 X$ *decomposes as* $\mathrm{Pic}^0 X \cong \ker f \times \ker(\mathrm{id} - f)$*. Moreover* $\dim \ker(\mathrm{id} - f) > 0$*.*
- *(b) Fix* $\bar{\beta} \in \ker f$ *such that* $U_0 \cap (\{\bar{\beta}\} \times \ker(\mathrm{id} - f))$ *is non-empty. Then, for* $\gamma \in U_0 \cap \ker(\mathrm{id} - f)$ *the line bundle* $\mathcal{O}_X(F_{\bar{\beta}\otimes\gamma}) \otimes \gamma^{-1}$ *does not depend on* γ*. Since it is effective by semicontinuity, we call it* $\mathcal{O}_X(F)$*.*
- *(c) For all* $(\beta, \gamma) \in \ker f \times \ker(\mathrm{id} - f) \cong \mathrm{Pic}^0 X$ *such that* $\beta \otimes \gamma \in U_0$*,* $|\mathcal{O}_X(F) \otimes \gamma|$ *is contained in the fixed divisor of* $\omega_X \otimes \beta \otimes \gamma$*.*

Proof. Let $\mathcal{O}_X(M_\alpha) = \omega_X \otimes a^*\alpha \otimes \mathcal{O}_X(-F_\alpha)$. Then, the proof of *(a)* is the same as [BLNP, Lem. 5.1]*(a)*. Item *(b)* follows directly from the definition of f. To prove *(c)*, let $(\beta, \gamma) \in \ker f \times \ker(\mathrm{id} - f)$

such that $\beta\otimes\gamma \in U_0$ and $E \in |\mathcal{O}_X(F)\otimes\gamma|$. Then $\mathcal{O}_X(F_{\beta\otimes\gamma}-E) \cong \mathcal{O}_X(F_{\beta\otimes\gamma}-F_{\bar\beta\otimes\gamma}) = f(\beta\otimes\bar\beta^{-1}) = \mathcal{O}_X$. Since $F_{\beta\otimes\gamma}$ is a fixed divisor of $|\omega_X\otimes\beta\otimes\gamma|$, also $E = F_{\bar\beta\otimes\gamma}$ is a fixed divisor in $|\omega_X\otimes\beta\otimes\gamma|$. $\square$

Using the decomposition given by the previous Lemma we give an explicit description of the "half" Poincaré line bundle.

Lemma 4.6 ([BLNP, Lem. 5.1, 5.3]). *We call $B = \mathrm{Pic}^0(\ker f)$ and $C = \mathrm{Pic}^0(\ker(\mathrm{id}-f))$ so that*

$$\mathrm{Alb}\, X \cong B\times C \qquad \text{and} \qquad \mathrm{Pic}^0 X \cong \mathrm{Pic}^0 B \times \mathrm{Pic}^0 C,$$

with $\dim C > 0$. Then we have the following description of "half" Poincaré line bundle.

$$(\mathrm{alb}\times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{O}_{B\times \mathrm{Pic}^0 B}\boxtimes \mathcal{P}_C) \cong \mathcal{O}_{X\times \mathrm{Pic}^0 X}(\overline{\mathcal{Y}})\otimes p^*\mathcal{O}_X(-F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(-\mathcal{D}_{\bar x}),$$

where $\bar x$ is such that $\mathrm{alb}(\bar x) = 0$ in $\mathrm{Alb}\, X$ and $\mathcal{P}_C$ is the Poincaré line bundle in $C\times \mathrm{Pic}^0 C$.

Proof. The decomposition of $\mathrm{Pic}^0 X$ comes directly from Lemma 4.5*(a)*. By the definition of $\overline{\mathcal{Y}}$ (see Proposition-Definition 4.4) and the definition of F (see Lemma 4.5*(b)*) we have that the line bundle

$$\mathcal{O}_{X\times \mathrm{Pic}^0 X}(\overline{\mathcal{Y}})\otimes p^*\mathcal{O}_X(-F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(-\mathcal{D}_{\bar p}),$$

- restricted to $X\times\{\beta\otimes\gamma\}$ is isomorphic to $\mathcal{O}_X(F_{\beta\otimes\gamma}-F) = \gamma$, for all $(\beta,\gamma)\in U_0 \subseteq \ker f \times \ker(\mathrm{id}-f)$;
- restricted to $\{\bar x\}\times \mathrm{Pic}^0 X$ is isomorphic to $\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar x})\otimes \mathcal{O}_{\mathrm{Pic}^0 X}(-\mathcal{D}_{\bar x})$, i.e. trivial.

On the other hand, $(\mathrm{alb}\times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{O}_{B\times \mathrm{Pic}^0 B}\boxtimes \mathcal{P}_C)$,

- restricted to $X\times\{\beta\otimes\gamma\}$ is isomorphic to γ, for all $(\beta,\gamma)\in \ker f\times \ker(\mathrm{id}-f)$;
- restricted to $\{\bar x\}\times \mathrm{Pic}^0 X$ is isomorphic to $\mathcal{O}_{\mathrm{Pic}^0 X}$, i.e. trivial.

Then, the Lemma follows from the see-saw principle. $\square$

5. The bicanonical map of irregular varieties

The next theorem gives a sufficient numerical condition for the birationality of the bicanonical map, analogous to Pareschi–Popa Theorem [PP2, Thm. 6.1] for the tricanonical map.

Theorem 5.1. *Let X be a smooth projective complex variety such that $\mathrm{gv}(\omega_X) \geq 2$. Then, the rational map associated to $\omega_X^2\otimes\alpha$ is birational onto its image for every $\alpha\in \mathrm{Pic}^0 X$.*

As a first corollary we have the following geometric interpretation.

Theorem 5.2. *Let X be a smooth projective complex variety of maximal Albanese dimension such that the bicanonical map is not birational. Then $0 \leq \mathrm{gv}(\omega_X) \leq 1$. Moreover, it admits a fibration onto a normal projective variety Y with $0 \leq \dim Y < \dim X$, any smooth model $\tilde Y$ of Y is of maximal Albanese dimension and*

$$q(X) - \dim X \leq q(\tilde Y) - \dim Y + \mathrm{gv}(\omega_X).$$

Proof. By Theorems 3.7 and 5.1, it is clear that $0 \leq \mathrm{gv}(\omega_X) \leq 1$. Now, the proof is the same as the proof of [PP3, Thm. B]. $\square$

Example 5.3. *We would like to show examples of varieties with $\mathrm{gv}(\omega_X) \geq 2$. For curves C, this is equivalent to $g(C) \geq 3$. For surfaces S, is equivalent to suppose that $q(S) \geq 4$ and S does not admit an irregular fibration to a curve of genus $\leq q(S) - 3$ (see [Be, Cor. 2.3]).*

On the other hand, if A is a simple abelian variety, then every subvariety X of codimension ≥ 2 has $\mathrm{gv}(\omega_X) \geq 2$. Moreover, the property of having $\mathrm{gv}(\omega_X) \geq 2$ is closed under taking products and cyclic coverings induced by a torsion point $\alpha \in \mathrm{Pic}^0 X - V^1(\omega_X)$.

The rest of the paper is devoted to the proof of Theorem 5.1.

Proof. Assume that $\mathrm{gv}(\omega_X) \geq 1$ and there exists $\alpha \in \mathrm{Pic}^0 X$ such that $\omega_X^{\otimes 2} \otimes \alpha$ is non-birational. Then, we want to see that $\mathrm{gv}(\omega_X) = 1$. Under these hypotheses we can apply Proposition-Definition 4.4 and Lemma 4.6, so $\mathrm{Alb}\, X \cong B \times C$, where $B = \mathrm{Pic}^0(\ker(\mathrm{id} - f))$ and $C = \mathrm{Pic}^0(\ker f)$. We have the following commutative diagram

$$\begin{tikzcd} \mathrm{Pic}^0 X \arrow[d, "p_{\hat b}"'] & X \times \mathrm{Pic}^0 X \arrow[l, "q"'] \arrow[r, "\mathrm{alb}\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times p_{\hat b}"] & \mathrm{Alb}\, X \times \mathrm{Pic}^0 X \arrow[d, "p_b \times p_{\hat b}"] \\ \mathrm{Pic}^0 B & X \times \mathrm{Pic}^0 B \arrow[l, "q"] \arrow[r, "b\times\mathrm{id}"'] & B \times \mathrm{Pic}^0 B \end{tikzcd} \tag{11}$$

where

- $p_b : \mathrm{Alb}\, X \to B$ and $p_{\hat b} : \mathrm{Pic}^0 X \to \mathrm{Pic}^0 B$ are the corresponding projections,
- b is the composition by $b : X \overset{\mathrm{alb}}{\to} \mathrm{Alb}\, X \overset{p_b}{\to} B$, and
- abusing notation we also call q either the projection $X \times \mathrm{Pic}^0 X \to \mathrm{Pic}^0 X$ or $X \times \mathrm{Pic}^0 B \to \mathrm{Pic}^0 B$ and p the projections $X \times \mathrm{Pic}^0 X \to X$ or $X \times \mathrm{Pic}^0 B \to X$.

The effectiveness of $\overline{\mathcal{Y}}$ give us the following short exact sequence on $X \times \mathrm{Pic}^0 X$

$$0 \to (\mathrm{alb}\times\mathrm{id})^*(\mathcal{O}_{B\times \mathrm{Pic}^0 B} \boxtimes \mathcal{P}_C)^{-1} \overset{\cdot\overline{\mathcal{Y}}}{\to} p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}(\mathcal{D}_{\bar x}) \to (p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}(\mathcal{D}_{\bar x}))|_{\overline{\mathcal{Y}}} \to 0.$$

Recall that $P = (\mathrm{alb}\times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B \boxtimes \mathcal{P}_C)$ since the Poincaré line bundle $\mathcal{P}$ in $\mathrm{Alb}\, X \times \mathrm{Pic}^0 X$ is isomorphic to $\mathcal{P}_B \boxtimes \mathcal{P}_C$. We apply the functor $R^d q_*(\,\cdot\, \otimes (\mathrm{alb}\times\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C}))$, that is, we tensor by the other "half" Poincaré line bundle and we consider the top direct image. We get

$$\cdots \to R^d\Phi_{P^{-1}}(\mathcal{O}_X) \to R^d q_*\left(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb}\times\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times\mathrm{Pic}^0 C})\right) \otimes \mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar x}) \to$$
$$\to R^d q_*\left((p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar x}))|_{\overline{\mathcal{Y}}} \otimes (\mathrm{alb}\times\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times\mathrm{Pic}^0 C})\right) \to 0$$

Using that $R^i\Phi_{P^{-1}} \cong (-1)^*_{\mathrm{Pic}^0 X} R^i\Phi_P$ (see (3)), we have the following short exact sequence,

$$0 \to (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \overset{\mu}{\to} \mathcal{E}(\mathcal{D}_{\bar x}) \to \mathcal{T} \to 0 \tag{12}$$

where:

(a) By base change, $\mathcal{E} = R^d q_*(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb}\times\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times\mathrm{Pic}^0 C}))$ is a coherent sheaf of rank $h^d(\mathcal{O}_X(F) \otimes \beta^{-1})$ by a general $\beta \in \ker f$, i.e $h^0(\omega_X \otimes \mathcal{O}_X(-F) \otimes \beta) = \chi(\omega_X)$ by Lemma 4.5*(c)*. Then,

$$\begin{aligned} \mathcal{E} &= R^d q_*(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb}\times\mathrm{id})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times\mathrm{Pic}^0 C})) & \\ &= R^d q_*(p^*\mathcal{O}_X(F) \otimes (\mathrm{id}\times p_{\hat b})^*(b\times\mathrm{id})^*\mathcal{P}_B^{-1}) & \text{right square of (11)} \\ &= R^d q_*(\mathrm{id}\times p_{\hat b})^*(p^*\mathcal{O}_X(F) \otimes (b\times\mathrm{id})^*\mathcal{P}_B^{-1}) & \text{abuse of notation on } p \\ &= p_{\hat b}^* R^d q_*(p^*\mathcal{O}_X(F) \otimes (b\times\mathrm{id})^*\mathcal{P}_B^{-1}) & \text{flat base change} \\ &= p_{\hat b}^* R^d\Phi_{P_b^{-1}}(\mathcal{O}_X(F)), & \end{aligned}$$

following the notation of (1) and (2).

(b) $\mathcal{T} = R^d q_*\left((p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar x}))|_{\overline{\mathcal{Y}}} \otimes (\mathrm{alb}\times\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times\mathrm{Pic}^0 C})\right)$ is supported at the locus of the $\alpha \in \mathrm{Pic}^0 X$ such that the fibre of the projection $q\colon \overline{\mathcal{Y}} \to \mathrm{Pic}^0 X$ has dimension d, i.e. it coincides with X. Such locus is contained in $V^1(\omega_X)$, therefore, since $\mathrm{gv}(\omega_X) \geq 1$, $\mathrm{codim\,supp}\,\mathcal{T} \geq 2$.

(c) The map μ is injective since it is a generically surjective map of sheaves of the same rank (recall that $\mathrm{rk}\,\widehat{\mathcal{O}_X} = \chi(\omega_X)$), and, as $\mathrm{gv}(\omega_X) \geq 1$, the source $\widehat{\mathcal{O}_X}$ is torsion-free (Thm. 3.3).

(d) μ is $R^d q_*(m_s)$, where m_s is the multiplication by the section defining $\overline{\mathcal{Y}}$. By base change [Mu, Cor. 3, pg. 53], $R^d q_*(m_s) \otimes \mathbb{C}(\alpha) = H^d(m_s|_{q^{-1}\{\alpha\}})$ where q is the projection $q\colon \overline{\mathcal{Y}} \to \mathrm{Pic}^0 X$. When $q^{-1}\{\alpha\} = X$, $m_s|_{q^{-1}\{\alpha\}} = 0$, so in these points $R^d q_*(m_s) \otimes \mathbb{C}(\alpha) = 0$.

Claim 5.4. $\mathcal{T} \neq 0$.

Proof of the Claim. Suppose that $\mathcal{T} = 0$, so μ is an isomorphism. Taking $\mathcal{E}xt^d(\,\cdot\,, \mathcal{O}_{\mathrm{Pic}^0 X})$ we get

$$\begin{aligned} k(\hat{0}) &= R^d\Phi_P\omega_X && \text{Prop. 3.6} \\ &= \mathcal{E}xt^d(\mathcal{E}, \mathcal{O}_{\mathrm{Pic}^0 X}) \otimes \mathcal{O}(-\mathcal{D}_{\bar{x}}) && \mathcal{E}xt^d(\mu, \mathcal{O}_{\mathrm{Pic}^0 X}) \text{ and Cor. 3.2} \\ &= p_{\hat{b}}^* \,\mathcal{E}xt^d(R^d\Phi_{P_b}(\mathcal{O}_X(F)), \mathcal{O}_{\mathrm{Pic}^0 B}) \otimes \mathcal{O}(-\mathcal{D}_{\bar{x}}) && \text{see item } \textit{(a)}, \end{aligned}$$

which implies that $\operatorname{codim}_{\mathrm{Alb}\,X} B = \dim\ker(\mathrm{id} - f) = 0$ contradicting Lemma 4.6. $\square$

Let $\tau(\mathcal{E}(\mathcal{D}_{\bar{x}}))$ be the torsion part of $\mathcal{E}(\mathcal{D}_{\bar{x}})$ and $\widetilde{\mathcal{E}(\mathcal{D}_{\bar{x}})}$ the quotient of $\mathcal{E}(\mathcal{D}_{\bar{x}})$ by its torsion part. Hence $\widetilde{\mathcal{E}(\mathcal{D}_{\bar{x}})}$ is torsion-free. Now consider the following composition

$$\begin{tikzcd} (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \arrow[r, "\mu"] \arrow[dr, "\tilde{\mu}"'] & \mathcal{E}(\mathcal{D}_{\bar{x}}) \arrow[d] \\ & \widetilde{\mathcal{E}(\mathcal{D}_{\bar{x}})}. \end{tikzcd}$$

Since $\tilde{\mu}$ is generically surjective and $(-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X}$ is torsion-free (recall that $\mathrm{gv}(\omega_X) \geq 1$), we have that $\tilde{\mu}$ is injective. Completing the diagram we get,

(13)
$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & \tau(\mathcal{E}(\mathcal{D}_{\bar{x}})) \arrow[r, equal] \arrow[d] & \tau(\mathcal{E}(\mathcal{D}_{\bar{x}})) \arrow[d] & \\ 0 \arrow[r] & (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \arrow[r, "\mu"] \arrow[d, equal] & \mathcal{E}(\mathcal{D}_{\bar{x}}) \arrow[r] \arrow[d] & \mathcal{T} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \arrow[r, "\tilde{\mu}"] & \widetilde{\mathcal{E}(\mathcal{D}_{\bar{x}})} \arrow[r] \arrow[d] & \widetilde{\mathcal{T}} \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

If $\widetilde{\mathcal{T}} = 0$, then the middle horizontal short exact sequence splits. But, for α a closed point in the support of $\mathcal{T}$ (by the previous claim we know that $\mathcal{T} \neq 0$), $\mu \otimes \mathbb{C}(\alpha) = 0$ by item *(d)*, so μ cannot split. Therefore $\widetilde{\mathcal{T}} \neq 0$.

Let $e = \operatorname{codim}_{\mathrm{Pic}^0 X} \operatorname{supp} \widetilde{\mathcal{T}} \geq 2$ (see item *(c)*). Then $\operatorname{codim}_{\mathrm{Pic}^0 X} \operatorname{supp} \mathcal{E}xt^e(\widetilde{\mathcal{T}}, \mathcal{O}_{\mathrm{Pic}^0 X}) = e$. Now, we apply the functor $\mathcal{E}xt^i(\,\cdot\,, \mathcal{O}_{\mathrm{Pic}^0 X})$ to the bottom row of (13) using Corollary 3.2

$$\ldots \to R^{e-1}\Phi_P\omega_X \to \mathcal{E}xt^e(\widetilde{\mathcal{T}}, \mathcal{O}_{\mathrm{Pic}^0 X}) \to \mathcal{E}xt^e(\widetilde{\mathcal{E}(\mathcal{D}_{\bar{x}})}, \mathcal{O}_{\mathrm{Pic}^0 X}) \to \ldots$$

Since $\widetilde{\mathcal{E}(\mathcal{D}_{\bar{x}})}$ is torsion-free, $\operatorname{codim}_{\mathrm{Pic}^0 X} \operatorname{supp} \mathcal{E}xt^e(\widetilde{\mathcal{E}(\mathcal{D}_{\bar{x}})}, \mathcal{O}_{\mathrm{Pic}^0 X}) > e$. Therefore, we must have $\operatorname{codim}_{\mathrm{Pic}^0 X} \operatorname{supp} R^{e-1}\Phi_P\omega_X = e$ and $\mathrm{gv}(\omega_X) \leq 1$. $\square$

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